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A resolution theorem for absolutely isolated singularities of holomorphic vector fields

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— Dedicated to the memory of Ricardo Mañé.

Abstract. In this paper, the desingularization problem for an absolutely isolated singularity of a n-dimensional holomorphic vector field is solved. Also, we exhibit final forms under blowing-up for this type of singularities.

0. Introduction

In this paper we solve the desingularization problem for an absolutely isolated singularity of a n-dimensional holomorphic vector field. Moreover, we exhibit final forms under blowing-up for this type of singularities with algebraic multiplicity one.

Let us give the precise statement of these results. Let $\mathcal{M}^n$ be a n-dimensional complex manifold. Let us consider a singular analytic foliation by curves on $\mathcal{M}^n$. By this we mean that at any point $p \in \mathcal{M}^n$ the foliation is generated by the holomorphic vector field

$$Z = \sum_{i=1}^{n} A_i \frac{\partial}{\partial z_i}, \; A_i \in \mathcal{O}_{n,p}; \; 1 \leq i \leq n \quad g.c.d.(A_1, \dots, A_n) = 1$$

where $\mathcal{O}_{n,p}$ is the ring of germs in p of analytic functions. In what follows we denote such a foliation by $\mathcal{F}_Z$ and the functions A_i are called *components* of Z.

The *algebraic multiplicity* $m_p(\mathcal{F}_Z)$ (or $m_p(Z)$), of $\mathcal{F}_Z$ at the point $p \in \mathcal{M}^n$, is the minimum of the orders $ord_p(A_i)$ (i.e., the order of the zero of A_i at p). We shall say that p is a *singular point* of $\mathcal{F}_Z$ if $m_p(Z) \geq 1$.

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The set of such points will be called $\mathrm{Sing}(\mathcal{F}_Z)$. A singular point $p \in \mathcal{M}^n$ is called *reduced* if $m_p(Z) = 1$ and the linear part of Z at p has at least one nonzero eigenvalue.

Let $E: \tilde{\mathcal{M}}^n \to \mathcal{M}^n$ be the blowing-up with center at the point $p \in \mathrm{Sing}(\mathcal{F}_Z)$. Then there exists a unique way of extending $E^*(\mathcal{F}_Z - \{p\})$ to a singular analytic foliation $\tilde{\mathcal{F}}_Z$ on a neighborhood of the projective space $\mathbb{C}P(n-1) = E^{-1}(p) \subset \tilde{\mathcal{M}}^n$, with singular set of codimension ≥ 2. In this case we say that $\tilde{\mathcal{F}}_Z$ is the *strict transform* of $\mathcal{F}_Z$ by E. We shall say that p is a *non-dicritical singularity* of $\mathcal{F}_Z$, when $E^{-1}(p)$ is invariant for $\tilde{\mathcal{F}}_Z$, i.e., it is the union of leaves and singularities of $\tilde{\mathcal{F}}_Z$. Otherwise p is called a *dicritical singularity.*

The desingularization problem for an isolated singularity $p \in \mathcal{M}^n$ (dicritical or not) of $\mathcal{F}_Z$ consists of proving the existence of a proper holomorphic map $\phi: \tilde{\mathcal{M}}^* \to \mathcal{M}^n$ of a n-dimensional complex manifold $\tilde{\mathcal{M}}^*$ such that:

a) $\phi^{-1}(p) = \bigcup_{i=1}^{N} D_i$; is a union of codimension one compact complex submanifolds with normal crossings.

b) The pull-back foliation $\phi^*(\mathcal{F}_Z|_{\mathcal{M}^n-(p)})$ extends to a singular foliation of $\tilde{\mathcal{M}}^*$ with singular set of codimension ≥ 2 and such that all singular points are reduced.

A first step towards the solution of the desingularization problem is to assume that the codimension of the singular set of the lifted foliation is n. This motivates the following:

Definition 1. *Let $\mathcal{F}_Z$ be an analytic foliation by curves on the n-dimensional complex manifold $\mathcal{M}^n$. We say that $p \in \mathrm{Sing}(\mathcal{F}_Z)$ is a absolutely isolated singularity (A.I.S.) of $\mathcal{F}_Z$ if and only if the following properties are verified:*

a) p is an isolated singularity of $\mathcal{F}_Z$,

b) let us denote $p = p_0$, $\mathcal{M}^n = \mathcal{M}_0^n$, $\mathcal{F}_Z = \mathcal{F}_0$, $\tilde{\mathcal{M}}^n = \mathcal{M}_1^n$, $\tilde{\mathcal{F}}_Z = \mathcal{F}_1$, $E_1 = E$. If we consider an arbitrary sequence of blowing-up's

$$\mathcal{M}_0^n \xleftarrow{E_1} \mathcal{M}_1^n \xleftarrow{E_2} \cdots \xleftarrow{E_N} \mathcal{M}_N^n$$

where the center of each E_i is a point $p_{i-1} \in \mathrm{Sing}(\mathcal{F}_{i-1})$ (here $\mathcal{F}_j$

denotes the strict transform of $\mathcal{F}_{j-1}$ *by* E_j, $1 \leq i,\ j \leq N$*), then* $\#\,\mathrm{Sing}(\mathcal{F}_N) < \infty$.

Observe that our definition of an absolutely isolated singularity is more general than the one given in [**C-C-S**] (this last will be called *non-dicritical absolutely isolated singularity*), in the sense that we are not excluding the case of dicritical singularities appearing in some step of the blowing-up process.

In this paper we prove the following desingularization result:

Theorem A. *Assume* $p \in \mathcal{M}^n$ *is an absolutely isolated singularity of* $\mathcal{F}_Z$*. Denote* $p = p_0$, $\mathcal{M}^n = \mathcal{M}_0^n$, $\mathcal{F}_Z = \mathcal{F}_0$, $E_1 = E$*. Then there exists a finite sequence of blowing-up's:*

$$\mathcal{M}_0^n \xleftarrow{E_1} \mathcal{M}_1^n \xleftarrow{E_2} \cdots \xleftarrow{E_N} \mathcal{M}_N^n$$

satisfying the following properties:

i) The center of each E_i *is a point* $p_{i-1} \in \mathrm{Sing}(\mathcal{F}_{i-1})$*, where* $\mathcal{F}_j$ *is the strict transform of the foliation* $\mathcal{F}_{j-1}$ *by* E_j, $(1 \leq i, j \leq N)$*,*

ii) if $q \in \mathrm{Sing}(\mathcal{F}_N)$*, then* q *is reduced.*

The main tool for proving this theorem is to use a formula relating the algebraic multiplicity of the original singularity to the Milnor numbers of the singularities which appear after a blowing-up. Observe that this program works at least when the set of the singularities at the projective space is isolated.

In dimension $n = 2$, it is well known that after finitely many 0 of blowing-ups at singular points, the foliation $\mathcal{F}_Z$ is transformed into a foliation $\mathcal{F}_Z^*$ with a finite number of singularities, all of them *simple* or *irreducible* and lying in the divisor (see [**C-L-S**], [**S**]). This means that if $p^* \in \mathrm{Sing}(\mathcal{F}_Z^*)$ then $\mathcal{F}_Z^*$ is locally generated by a vector field Z^* having a linear part with eigenvalues 1 and λ, where $\lambda \notin \mathbb{Q}_+$ ($\mathbb{Q}_+$: strictly positive rational numbers).

The simple singularities may be thought of a *final forms* in the sense that they are persistent under new blowing-ups. The local topological structure of these singularities has been studied by several authors (see [**C**], [**M-N**]).

In [**C-C-S**], the authors extend the concept of simple singularity (or irreducible singularity) to n-dimensional case, provided that the singularity is absolutely isolated non-dicritical (i.e., do not appear dicritical singularities in the blowing-up process). Here, we will prove that if p is a reduced non-dicritical singularity of the foliation $\mathcal{F}_Z$ such that p is an A.I.S. then p is an absolutely isolated non-dicritical singularity, and so we can apply the results in [**C-C-S**].

It must be mentioned that final forms for a three-dimensional vector field were given by Cano in [**Ca1**].

The desingularization problem, when $n = 2$, was studied by I. Bendixson [**B**] and by H. Dulac [**D**] at the beginning of this century. It was solved by A. Seidenberg [**S**] in the sixties. Another proof was given by A. Ven Den Essen [**V**], his arguments use the concept of multiplicity of intersection between analytic curves. A strategy for the general three-dimensional case was developed by F. Cano [**Ca2**]; however a definite result is still missing.

We have to mention that, in the n-dimensional case, the unique known result was obtained by C. Camacho, F. Cano and P. Sad [**C-C-S**]. In this reference, the authors assume that p is a non-dicritical absolutely isolated singularity generalizing the methods given by C. Camacho and P. Sad in [**C-S**] when $n = 2$.

This paper is organized as follows: In section 1, we recall some elementary properties about blowing-up's and we prove a formula relating the Milnor number of a dicritical singularity with the algebraic multiplicity of the singularity and the Milnor numbers of the singularities of the strict transform. The section 2 is devoted to solve the desingularization problem for an A.I.S. Finally, in section 3 we study the final forms for reduced and absolutely isolated singularity of a foliation by curves.

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1. The Milnor Number of an Isolated Dicritical Singularity

Let $\mathcal{O}_{n,p}$ be the ring of germs at $p \in U \subset \mathbb{C}^n$ of holomorphic functions and let $I[A_1, \dots, A_n] \subset \mathcal{O}_{n,p}$ be the ideal generated by the components of a holomorphic vector field Z. We define the *Milnor number* $\mu_p(Z)$ of Z at p, as

$$\mu_p(Z) = \dim_{\mathbb{C}} \left(\frac{\mathcal{O}_{n,p}}{I[A_1, \dots, A_n]} \right) \tag{1.1}$$

This number is finite if and only if p is an isolated singularity of Z, and $\mu_p(Z) = 0$ if and only if p is a regular point of Z (see [**G-H**]).

The Milnor number again can be geometrically interpreted as the *intersection index* $i_o(A_1, \dots, A_n)$ at p of the n analytic hypersurface generated by the components of Z (see [**Ch**]):

$$\mu_p(Z) = i_p(A_1, \dots, A_n) \tag{1.2}$$

Let $p \in U$ be an isolated singularity of the vector field Z, such that $m_p(Z) = \nu$ and $\mathcal{F}_Z$ the foliation generated by Z. Let $\tilde{\mathcal{F}}_Z$ be the strict transform of $\mathcal{F}_Z$, which is generated by $\tilde{Z}$. When $n = 2$, there exists a formula relating ν to the Milnor number of Z at p and the Milnor numbers of the singularities of $\tilde{Z}$ (see [**M-M**]): $\mu_p(Z)$ is given by

$$\mu_p(Z) = \begin{cases} \nu^2 - \nu - 1 + \sum\limits_{q \in E^{-1}(p)} \mu_q(\tilde{Z}), \\ \qquad\qquad \text{if } P \text{ is a non-dicritical singularity,} \\ \nu^2 + \nu - 1 + \sum\limits_{q \in E^{-1}(p)} \mu_q(\tilde{Z}), \\ \qquad\qquad \text{if } p \text{ is a dicritical singularity.} \end{cases} \tag{1.3}$$

Since $\#\,\mathrm{Sing}(\tilde{Z}) < \infty$, the sums in (1.3) are finite. There exists a n-dimensional generalization of (1.3) in the case that p is an isolated non-dicritical singularity of Z, provided that $\#\,\mathrm{Sing}(\tilde{\mathcal{F}}_Z) < \infty$ (see [**C-C-S**]):

$$\mu_p(Z) = \nu^n - \nu^{n-1} - \dots - \nu - 1 + \sum_{q \in E^{-1}(p)} \mu_q(\tilde{Z}) \tag{1.4}$$

This section is devoted to the proof of an analogous formula to (1.4) in the case that p is an isolated dicritical singularity of Z such that $\#\,\mathrm{Sing}(\tilde{\mathcal{F}}_Z) < \infty$. Before proving this formula, let us recall some elementary facts about blowing-up's.

Let $\mathcal{M}^n$ be a n-dimensional complex manifold and let us consider an analytic foliation by curves $\mathcal{F}_Z$ on $\mathcal{M}^n$. Suppose that $p \in \mathcal{M}^n$ is an isolated singularity of $\mathcal{F}_Z$. Let $z = (z_1, \dots, z_n)$ be local coordinates of a neighborhood U of p in $\mathcal{M}^n$ such that $p = (0, \dots, 0) \in \mathbb{C}^n$. In these coordinates, $\mathcal{F}_Z$ is generated by the holomorphic vector field $Z = \sum_{i=1}^{n} A_i \frac{\partial}{\partial z_i}$, and if $m_0(Z) = \nu (\nu \in \mathbb{Z}^+)$, then the components A_i of Z have a Taylor development at $0 \in \mathbb{C}^n$

$$A_i = \sum_{k \geq \nu} A_k^i, \ 1 \leq i \leq n \tag{1.5}$$

where each A_k^i are homogeneous polynomials of degree k.

For each $j = 1, \dots, n$ we define $U_j = \{(z_1, \dots, z_n) \in \mathbb{C}^n : z_j \neq 0\}$ and $\tilde{U}_j = E^{-1}[U_j]$, where E is the blowing-up with center at $0 \in \mathbb{C}^n$. In $\tilde{U}_j$ we introduce coordinates $y = (y_1, \dots, y_n)$ and E has the following expression:

$$E(y_1, \dots, y_n) = (z_1, \dots, z_n); \text{ where } y_j = z_j \text{ and } y_i = z_i/z_j \text{ if } i \neq j \tag{1.6}$$

and

$$E^{-1}(0) \cap \tilde{U}_j = \{(y_1, \dots, y_n) \in \tilde{U}_j : y_j = 0\} \tag{1.7}$$

In this chart, the pull-back of Z by E is generated by:

$$E^* Z = A_j \circ E \frac{\partial}{\partial y_j} + \sum_{\substack{i=1 \\ i \neq j}}^{n} \left(\frac{A_i \circ E - y_i A_j \circ E}{y_j} \right) \frac{\partial}{\partial y_i} \tag{1.8}$$

From (1.5) and (1.8):

$$\begin{aligned} E^* Z(y) = & \left(\sum_{k \geq \nu} y_j^k A_k^j(\hat{y}) \right) \frac{\partial}{\partial y_j} \\ & + \sum_{\substack{i=1 \\ i \neq j}}^{n} \left(\sum_{k \geq \nu} y_j^{k-1} [A_k^i(\hat{y}) - y_i A_k^j(\hat{y})] \right) \frac{\partial}{\partial y_i} \end{aligned} \tag{1.9}$$

The following result shows that the condition of $\mathcal{F}_Z$ has a dicritical singularity in $0 \in \mathbb{C}^n$ can be characterized in terms of the polynomials $A_\nu^i (1 \leq i \leq n)$, i.e., of $J_0^\nu(Z)$: the jet of order ν of Z at the origin.

Proposition 1. *With the above notations, the following assertions are equivalent:*

a) $0 \in \mathbb{C}^n$ *is a dicritical singularity of* $\mathcal{F}_Z$.

b) $z_j A^i_\nu - z_i A^j_\nu = 0;\ \forall 1 \leq i < j \leq n.$

c) $J^\nu_0(Z) = P_{\nu-1}R$, *where* $R = \sum\limits_{i=1}^{n} z_i \frac{\partial}{\partial z_i}$ *is the radial vector field and* $P_{\nu-1}$ *is a homogeneous polynomial of degree* $\nu - 1$.

The proof of Proposition 1 is not difficult and it is left to the reader.

Remark. If p is a dicritical singularity of $\mathcal{F}_Z$ and $P_{\nu-1}$ is the polynomial of Proposition 1, then we can define the following algebraic hypersurface on $\mathbb{C}P(n-1)$

$$S = \{[z_1; \dots ; z_n] \in \mathbb{C}P(n-1) \colon P_{\nu-1}(z_1, \dots , z_n) = 0\}$$

It is not difficult to see that $\mathrm{Sing}(\tilde{\mathcal{F}}_Z) \subseteq S$ and if $\tilde{p} \in S - \mathrm{Sing}(\tilde{\mathcal{F}}_Z)$ then the leaf of $\tilde{\mathcal{F}}_Z$ through $\tilde{p}$ is tangent to the projective space $E^{-1}(0)$.

Returning to the initial problem, we have the following result:

Theorem 1. *Let* Z *be a holomorphic vector field with isolated singularity at* $0 \in \mathbb{C}^n$ *such that* $\tilde{Z}$ *has isolated singularities. If* $0 \in \mathbb{C}^n$ *is a dicritical singularity and* $m_0(Z) = \nu$, *then*

$$\mu_0(Z) = g(\nu + 1) + \sum_{q \in E^{-1}(0)} \mu_q(\tilde{Z}),$$

where $g(\nu) = \nu^n - \nu^{n-1} - \dots - \nu - 1$.

Proof. Let $Z = \sum\limits_{k \geq \nu} Z_k$ where $Z_\nu = P_{\nu-1} \sum\limits_{i=1}^{n} z_i \frac{\partial}{\partial z_i}$. We consider the vector field $Z_{\nu+1} + R$ (with $R = \sum\limits_{k \geq \nu+2} Z_k$) and we suppose that:

a) $Z_{\nu+1} + R$ has isolated singularity at $0 \in \mathbb{C}^n$ and

b) the strict transform $\tilde{Z}_{\nu+1} + \tilde{R}$ has isolated singularities at the divisor $E^{-1}(0)$.

It is easy to see that $0 \in \mathbb{C}^n$ is a non-dicritical isolated singularity of the vector field $Z_{\nu+1} + R$, thus from (1.4) we have that:

$$\mu_0(Z_{\nu+1} + R) = g(\nu + 1) + \sum_{\tilde{q} \in E^{-1}(0)} \mu_{\tilde{q}}(\tilde{Z}_{\nu+1} + \tilde{R}) \tag{1.10}$$

where $g(\nu) = \nu^n - \nu^{n-1} - \dots - \nu - 1$.

From the hypothesis b) we can suppose, without loss of generality, that the singularities of $\tilde{Z}_{\nu-1} + \tilde{R}$ are in the chart $\tilde{U}_1$ of $\tilde{\mathbb{C}}^n$. Therefore

$$E^*[Z_{\nu+1} + R](y) = \left(\sum_{k \geq \nu+1} y_1^k A_k^1(\hat{y}) \right) \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \left(\sum_{k \geq \nu+1} y_1^{k-1} [A_k^i(\hat{y}) - y_i A_k^1(\hat{y})] \right) \frac{\partial}{\partial y_i}$$

where $y = (y_1, \dots, y_n)$ and $\hat{y} = (1, y_2, \dots, y_n)$. Thus, $E^*[Z_{\nu+1} + R]$ is divisible by y_1^ν and we have that:

$$\tilde{Z}_{\nu+1}(y) + \tilde{R}(y) = y_1 A_{\nu+1}^1(\hat{y}) \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \left(A_{\nu+1}^i(\hat{y}) - y_i A_{\nu+1}^1(\hat{y}) \right) \frac{\partial}{\partial y_i} + y_i \tilde{R}(y) \tag{1.11}$$

We conclude that the singularities of $\tilde{Z}_{\nu+1} + \tilde{R}$ are the points $\tilde{q}_j = (0, y_2^j, \dots, y_n^j)$, $1 \leq j \leq N$, where $y_2^j, \dots, y_n^j$ satisfies the following conditions:

$$\begin{aligned} &A_{\nu+1}^i(1, y_2^j, \dots, y_n^j) - y_i^j A_{\nu+1}^1(1, y_2^j, \dots, y_n^j) \\ &= 0, \ 2 \leq i \leq n, \ i \leq j \leq N \end{aligned} \tag{1.12}$$

For $\epsilon > 0$, we consider the perturbation $Z_\epsilon = \epsilon Z_\nu + Z_{\nu+1} + R$. Clearly $0 \in \mathbb{C}^n$ is a dicritical isolated singularity of Z_ϵ and $E^*(Z_\epsilon)$ is divisible by y_1^ν. We have that

$$\tilde{Z}_\epsilon(y) = \epsilon P_{\nu-1}(\hat{y}) \frac{\partial}{\partial y_1} + \tilde{Z}_{\nu+1}(y) + \tilde{R}(y) \tag{1.13}$$

or equivalently:

$$\tilde{Z}_\epsilon(y) = \left[\epsilon P_{\nu-1}(\hat{y}) + y_1 A_{\nu+1}^1(\hat{y}) \right] \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \left(A_{\nu+1}^i(\hat{y}) - y_i A_{\nu+1}^1(\hat{y}) \right) \frac{\partial}{\partial y_i} + y_1 \tilde{R}(y) \tag{1.14}$$

Then, we have two kinds of singularities of $\tilde{Z}_\epsilon$:

- Singularities inside the divisor;
- Singularities outside the divisor.

Singularities inside the divisor are the points

$$\tilde{p}_j = (0, y_2^j, \dots, y_n^j)$$

where $y_2^j, \dots, y_n^j$ satisfy the conditions (1.12) and

$$P_{\nu-1}(1, y_2^j, \dots, y_n^j) = 0.$$

Then there exists $0 \leq N_i < N$ such that $\tilde{p}_j = \tilde{q}_j$, $\forall\ 1 \leq j \leq N_1$. Observe that these points are also singularities of $\tilde{Z}$.

Singularities outside the divisor are the points

$$\tilde{p}_k(\epsilon) = (y_1^k(\epsilon), \dots, y_n^k(\epsilon))$$

with $y_1^k(\epsilon) \neq 0$. From (1.13), it follows that

$$\lim_{\epsilon \to 0} \tilde{p}_k(\epsilon) = \tilde{q}_j,\ \forall\ k \in I_j,\ \forall\ 1 \leq j \leq N;$$

where I_j is a finite set of indices.

For each singularity $\tilde{p}_k = \tilde{p}_k(\epsilon)$ of $\tilde{Z}_\epsilon$ outside the divisor, we denote $p_k = E(\tilde{p}_k)$ and so:

$$\mu_{p_k}(Z_\epsilon) = \mu_{\tilde{p}_k}(\tilde{Z}_\epsilon) \tag{1.15}$$

If ϵ is small enough, it follows that:

$$\mu_{\tilde{q}_j}(\tilde{Z}_{\nu+1} + \tilde{R}) = \begin{cases} \mu_{\tilde{p}_j}(\tilde{Z}_\epsilon + \sum\limits_{k \in I_j} \mu_{\tilde{p}_k}(\tilde{Z}_\epsilon), & \text{if } 1 \leq j \leq N_1 \\ \sum\limits_{k \in I_j} \mu_{\tilde{p}_k}(\tilde{Z}_\epsilon), & \text{if } N_1 + 1 \leq j \leq N \end{cases} \tag{1.16}$$

and

$$\mu_0(Z_{\nu+1} + R) = \mu_0(Z_\epsilon) + \sum_{j=1}^{N} \sum_{k \in I_j} \mu_{p_k}(Z_\epsilon) \tag{1.17}$$

From (1.10), (1.15), (1.16) and (1.17), we have that:

$$g(\nu+1) + \sum_{j=1}^{N} \mu_{\tilde{q}_j}(\tilde{Z}_{\nu+1} + \tilde{R}) = \mu_0(Z_\epsilon) + \sum_{j=1}^{N} \sum_{k \in I_j} \mu_{\tilde{p}_k}(\tilde{Z}_\epsilon) =$$

$$= \mu_0(Z_\epsilon) + \sum_{j=1}^{N} \mu_{\tilde{q}_j}(\tilde{Z}_{\nu+1} + \tilde{R}) - \sum_{j=1}^{N_1} \mu_{\tilde{p}_j}(\tilde{Z}_\epsilon)$$

Thus:

$$\mu_0(Z_\epsilon) = g(\nu+1) + \sum_{\tilde{q}\in E^{-1}(0)} \mu_{\tilde{q}}(\tilde{Z}_\epsilon) \tag{1.18}$$

Now, we assert that $\mu_0(Z) = \mu_0(Z_\epsilon)$ and $\mu_{\tilde{p}_j}(\tilde{Z}) = \mu_{\tilde{p}_j}(\tilde{Z}_\epsilon)$, $1 \leq j \leq N_1$. In fact, since $Z_\nu(0) = 0$, there exists $r > 0$ such that

$$||Z|| < r \Rightarrow ||Z(z) - Z_\epsilon(z)|| = (1-\epsilon)||Z_\nu(z)|| < 2.$$

Let $0 < r_1 < r$ and consider the homotopy $G: [0,1] \times S^{2n-1}_{r_1} \to S^{2n-1}$ given by

$$G(t,z) = \frac{tZ_\epsilon(z) + (1-t)Z(z)}{||tZ_\epsilon(z) + (1-t)Z(z)||},$$

then $G(0,z) = Z(z)/||Z(z)||$ and $G(1,z) = Z_\epsilon(z)/||Z_\epsilon(z)||$, hence $\mu_0(Z) = \mu_0(Z_\epsilon)$.

Let $\tilde{p}_j = (0, y_2^j, \dots, y_n^j)$ $(1 \leq j \leq N_1)$ a singularity of $\tilde{Z}_\epsilon$ (it is also singularity of $\tilde{Z}$), then $P_{\nu-1}(1, y_2^j, \dots, y_n^j) = 0$. It follows that there exists $\tilde{r} > 0$ such that

$$||(y_2, \dots, y_n) - (y_2^j, \dots, y_n^j)|| < \tilde{r} \Rightarrow |P_{\nu-1}(1, y_2, \dots, y_n)| < \frac{2}{1-\epsilon}.$$

Thus from (1.14), we have that

$$||y - \tilde{p}_j|| < \tilde{r} \Rightarrow ||\tilde{Z}(y) - \tilde{Z}_\epsilon(y)|| = (1-\epsilon)|P_{\nu-1}(1, y_2, \dots, y_n)| < 2.$$

Let $0 < \tilde{r}_1 < \tilde{r}$ and consider the homotopy $\tilde{G}: [0,1] \times S^{2n-1}_{\tilde{r}_1}(\tilde{p}_j) \to S^{2n-1}$, $(S^{2n-1}_{\tilde{r}_1}(\tilde{p}_j) = \{||y - \tilde{p}_j|| = \tilde{r}_1\})$ given by

$$\tilde{G}(t,y) = \frac{t\tilde{Z}_\epsilon(y) + (1-t)\tilde{Z}(y)}{||t\tilde{Z}_\epsilon(y) + (1-t)\tilde{Z}(y)||},$$

then then $\tilde{G}(0,y) = \tilde{Z}(y)/||\tilde{Z}(y)||$ and $\tilde{G}(1,z) = \tilde{Z}_\epsilon(z)/||\tilde{Z}_\epsilon(z)||$, hence $\mu_{\tilde{p}_j}(\tilde{Z}) = \mu_{\tilde{p}_j}(\tilde{Z}_\epsilon)$, $\forall\ 1 \leq j \leq N_1$, and so the assertion is proved.

From (1.18) it follows that

$$\mu_0(Z) = g(\nu+1) + \sum_{\tilde{q}\in E^{-1}(0)} \mu_{\tilde{q}}(\tilde{Z}) \tag{1.19}$$

In the case that $0 \in \mathbb{C}^n$ is not isolated singularity of $Z_{\nu+1} + \mathbb{R}$, we consider the perturbation $Z_\delta = Z_\nu + Z_{\nu+1} + \delta Y_{\nu+1} + R$ where $Y_{\nu+1}$ is a homogeneous vector field of degree $\nu+1$ such that $0 \in \mathbb{C}^n$ is an isolated singularity of $Z_{\nu+1} + \delta Y_{\nu+1}$. Observe that if $\delta > 0$ is small enough then $0 \in \mathbb{C}^n$ is an isolated singularity of Z_δ. In fact, since that $Y_{\nu+1}(0) = 0$, there exists $r > 0$ such that $\|z\| < r \Rightarrow \|Y_{\nu+1}(z)\| < 1$. As $0 \in \mathbb{C}^n$ is an isolated singularity of Z, then we define $m = \inf\{\|Z(z)\|: \|z\| = r'\}$, where $0 < r' < r$. Thus $\|Z_\delta(z)\| \geq \|Z(z)\| - \delta\|Y_{\nu+1}(z)\| > m - \delta$. Therefore, if $\delta < m$ then $\|Z_\delta(z)\| > 0$, $\forall \|z\| = r'$, hence $0 \in \mathbb{C}^n$ is isolated singularity of Z_δ. Therefore, the vector field Z_δ has dicritical isolated singularity in $0 \in \mathbb{C}^n$, and satisfies the conditions a), b) above. From (1.19):

$$\mu_0(Z_\delta) = g(\nu+1) + \sum_{\tilde{q} \in E^{-1}(0)} \mu_{\tilde{q}}(\tilde{Z}_\delta) \tag{1.20}$$

As before, we can to prove that $\mu_0(Z) = \mu_0(Z_\delta)$ and $\mu_{\tilde{p}}(\tilde{Z}) = \mu_{\tilde{p}}(\tilde{Z}_\delta)$. This finishes the proof of Theorem 1.

2. The Theorem of Desingularization

This section is devoted to the proof of Theorem A. Notice that, by Theorem 1 and the formula (1.4), for p singularity of the vector field Z with $m_p(Z) = \nu$, we can write

$$\mu_p(Z) = g(\sigma) + \sum_{\tilde{p} \in E^{-1}(p)} \mu_q(\tilde{Z}) \tag{2.1}$$

where $g(\sigma) = \sigma^n - \sigma^{n-1} - \cdots - \sigma - 1$, with $\sigma = \nu$ if p is a non-dicritical singularity of Z and $\sigma = \nu + 1$, otherwise. It is not difficult to see that the function g is an increasing function for all $\sigma \geq 2$.

Theorem A. *Assume $p \in \mathcal{M}^n$ is an absolutely isolated singularity of $\mathcal{F}_Z$. Denote $p = p_0$, $\mathcal{M}^n = \mathcal{M}^n_0$, $\mathcal{F}_Z = \mathcal{F}_0$, $E_1 = E$. Then there exists a finite sequence of blowing-up's:*

$$\mathcal{M}^n_0 \xleftarrow{E_1} \mathcal{M}^n_1 \xleftarrow{E_2} \cdots \xleftarrow{E_N} \mathcal{M}^n_N$$

satisfying the following properties:

i) The center of each E_i is a point $p_{i-1} \in \mathrm{Sing}(\mathcal{F}_{i-1})$, where $\mathcal{F}_j$ is the strict transform of the foliation $\mathcal{F}_{j-1}$ by E_j, $(1 \leq i, j \leq N)$;

ii) if $q \in \mathrm{Sing}(\mathcal{F}_N)$, then q is reduced.

Proof. Suppose that $m_p(Z) = \nu > 1$. Since p is an A.I.S. of $\mathcal{F}_Z$, from (2.1), we obtain that

$$\mu_{\tilde{p}}(\tilde{Z}) < \mu_p(Z);\ \forall\, \tilde{p} \in E^{-1}(p).$$

Since $\mu_p(Z) \geq m_p(Z)$, $\forall p$; after a finite number of successive blowing-up's $E_1 = E, E_2, \dots, E_N$ with centers at singular points, we will obtain only points with algebraic multiplicity ≤ 1.

We define $\phi = E_N \circ E_{N-1} \circ \cdots \circ E$, it follows that $\phi\colon \mathcal{M}^n_N \to \mathcal{M}^n_0$ is a proper holomorphic map and the pull-back $\phi^*(\mathcal{F}_0|_{\mathcal{M}^n - \{p\}})$ extends to a singular foliation $\mathcal{F}_N$ on $\mathcal{M}^n_N$ with singular set of codimension n.

Thus, if $q \in \mathrm{Sing}(\mathcal{F}_N)$ then $m_q(\mathcal{F}_N) = 1$. The Theorem A is a consequence of the following:

Lemma 1. *Let $p \in \mathcal{M}^n (n \geq 3)$ be a singular point of $\mathcal{F}_Z$ such that $m_p(Z) = 1$ and p is not reduced. Then p is not an A.I.S.*

Proof of Lemma 1. Let $z = (z_1, \dots, z_n)$ be local coordinates of a neighborhood of p in $\mathcal{M}^n$ such that $p = (0, \dots, 0) \in \mathbb{C}^n$. In these coordinates, $\mathcal{F}_Z$ is generated by the holomorphic vector field $Z = \sum\limits_{i=1}^{n} A_i \frac{\partial}{\partial z_i}$, where $A_i = \sum\limits_{k \geq \nu} A^i_k$ and A^i_k are homogeneous polynomials of degree k. Since p is not a reduced singular point, by the Jordan canonical form we have that

$$Z(z) = (z_2 + \sum_{k\geq 2} A^1_k(z))\frac{\partial}{\partial z_1} + (\epsilon_i z_{i+1} + \sum_{k \geq 2} A^i_k(z))\frac{\partial}{\partial z_i} + \left(\sum_{k\geq 2} A^n_k(z)\right)\frac{\partial}{\partial z_n}$$

where $\epsilon_j \in \{0, 1\}$, $\forall\ j = 1, \dots, n-1$. In the chart of the blowing-up $y_1 = z_1, y_i = z_i/z_1 (2 \leq i \leq n)$, we have that the strict transform $\tilde{\mathcal{F}}_Z$ is generated by $\tilde{Z} = \sum\limits_{i=1}^{n} \tilde{A}_i \frac{\partial}{\partial y_i}$, where:

$$\tilde{A}_1(y) = y_1 y_2 + \sum_{k\geq 2} A^1_k(\hat{y}) y_1^k$$

$$\tilde{A}_i(y) = \epsilon_i y_{i+1} - y_2 y_i + \sum_{k \geq 2} [A^i_k(\hat{y}) - y_i A^1_k(\hat{y})] y_1^{k-1},\ 2 \leq i \leq n-1$$

$$\tilde{A}_n(y) = -y_2 y_n + \sum_{k\geq 2} [A^n_k(\hat{y}) - y_n A^1_k(\hat{y})] y_1^{k-1}$$

with $\hat{y} = (1, y_2, \ldots, y_n)$. Thus

$$\tilde{Z}(0, y_2, \ldots, y_n) = \sum_{i=2}^{n-1} (\epsilon_i y_{i+1} - y_2 y_i) \frac{\partial}{\partial y_i} - y_2 y_n \frac{\partial}{\partial y_n}.$$

Now, we consider two cases:

Case 1: There exists $i_0 \in \{2, \ldots, n-1\}$ such that $\epsilon_{i_0} = 0$. In this case:

$$\tilde{Z}(0, y_2, \ldots, y_n) = \sum_{\substack{i=2 \\ i \neq i_0}}^{n-1} (\epsilon_i y_{i+1} - y_2 y_i) \frac{\partial}{\partial y_i} - y_2 y_{i_0} \frac{\partial}{\partial y_{i_0}} - y_2 y_n \frac{\partial}{\partial y_n}$$

It is easy to see that $\tilde{Z}(0, \ldots, 0, y_{i_0+1}, 0, \ldots, 0) = 0, \ \forall \ y_{i_0+1} \in \mathbb{C}$, therefore $\# \operatorname{Sing}(\tilde{\mathcal{F}}_Z) = \infty$, and so p is not an A.I.S.

Case 2: $\epsilon_2 = \cdots = \epsilon_{n-1} = 1$. In this case it is not difficult to see that $0 \in \mathbb{C}^n$ in the chart $y_1 = z_1$, $y_i = z_i/z_2 (2 \leq i \leq n)$, is the unique singularity of $\mathcal{F}_Z$, moreover:

$$D\tilde{Z}(0) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ \alpha_2 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha_{n-1} & 0 & 0 & \ldots & 0 & 1 \\ \alpha_n & 0 & 0 & \ldots & 0 & 0 \end{pmatrix} \tag{2.2}$$

where $\alpha_i = A_2^i(1, 0, \ldots, 0), 2 \leq i \leq n$.

The characteristic polynomial of $D\tilde{Z}(0)$ is $\Delta(t) = t^n$. In order to obtain the Jordan canonical form of $D\tilde{Z}(0)$, we shall compute the minimum polynomial $m(t)$ of $D\tilde{Z}(0)$. Observe that:

$$\tilde{M} = D\tilde{Z}(0) = \begin{pmatrix} 0 & \Theta \\ P & R_{n-1}(1) \end{pmatrix} \tag{2.3}$$

where $0 \in \mathbb{C}^{1\times 1} \approx \mathbb{C}$, $\Theta = [0 \ldots 0] \in \mathbb{C}^{1\times(n-1)} \approx \mathbb{C}^{n-1}$, $P^t = [\alpha_2 \ldots \alpha_n] \in \mathbb{C}^{1\times(n-1)} \approx \mathbb{C}^{n-1}$ and $R_{n-1}(1) \in \mathbb{C}^{(n-1)\times(n-1)}$ is the upper triangular matrix of order one. (Here $\mathbb{C}^{n\times m}$ denotes the matrix space of n rows and m columns). We will denote $R_{n-1}(k) = [R_{n-1}(1)]^k$, $\forall \ k \in \mathbb{Z}^+$. Under these notations, it is not difficult to prove that:

$$\tilde{M}^k = \begin{pmatrix} 0 & \Theta \\ R_{n-1}(k-1)P & R_{n-1}(k) \end{pmatrix} \ \forall \ k \in \mathbb{Z}^+ \tag{2.4}$$

Since $R_{n-1}(k)=0$ if and only if $k \geq n-1$ we have that $\tilde{M}^k \neq 0$, $\forall 1 \leq k \leq n-2$. Observe that

$$\tilde{M}^{n-1} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \alpha_n & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

thus, we have two possibilities:

i) If $\alpha_n = 0$ then $M^{n-1}=0$, and so $m(t)=t^{n-1}$. Therefore $D\tilde{Z}(0)$ has Jordan canonical form (4.13) with $\epsilon_2 = \dots = \epsilon_{n-2} = 1$ and $\epsilon_{n-1}=0$. Thus 0 is not an A.I.S. of $\mathcal{F}_Z$.

ii) If $\alpha_n \neq 0$ then we affirm that there exists a linear change of coordinates φ such that $\varphi^*\tilde{Z}$ satisfies the conditions in Case 2-(i). In fact, we define the linear maps $\varphi = (\varphi_1, \dots, \varphi_n)\colon \mathbb{C}^n \to \mathbb{C}^n$ and $\psi = (\psi_1, \dots, \psi_n)\colon \mathbb{C}^n \to \mathbb{C}^n$, where:

$$\varphi_1(x) = \frac{1}{\alpha_n}x_n,\ \varphi_2(x) = x_1 \text{ and } \varphi_i(x) = x_{i-1} - \frac{\alpha_{i-1}}{\alpha_n}x_n \ (3 \leq i \leq n)$$
$$\psi_1(y) = y_2,\ \psi_i(y) = \alpha_i y_1 + y_{i+1} \ (2 \leq i \leq n-1) \text{ and } \psi_n(y) = \alpha_n y_1.$$

It is clear that $\psi = \varphi^{-1}$. Now, we define $X = \varphi^*\tilde{Z} = \psi\tilde{Z}\circ\varphi$. If we denote $X = \sum_{i=1}^{n} B_i \frac{\partial}{\partial x_i}$, then $B_1 = \tilde{A}_2 \circ \varphi$, $B_i = \alpha_i \tilde{A}_1 \circ \varphi + \tilde{A}_{i+1} \circ \varphi (2 \leq i \leq n-1)$ and $B_n = \alpha_n \tilde{A}_1 \circ \varphi$. Since $DX(0) = \psi D\tilde{Z}(0)\varphi$; an easy computation shows that $DX(0) = R_n(1)$, moreover, in the chart $u_1 = x_1$, $u_i = x_i/x_1 (2 \leq i \leq n)$; we have that $D\tilde{X}(0)$ is like (2.3) with $P^t = [\beta_2 \cdots \beta_n]$, where $\beta_i = B_2^i(1,0,\dots,0)$, $2 \leq i \leq n$. Note that:

$$\beta_n = B_2^n(1,0,\dots,0) = \frac{\partial^2 B_n}{\partial x_1^2}(0,\dots,0) = \alpha_n \frac{\partial^2 \tilde{A}_1}{\partial y_2^2}(0,\dots,0) = 0.$$

Thus 0 is not an A.I.S. of $X = \varphi^*\tilde{Z}$. This finishes the proof of Lemma 1.

3. Reduction of Singularities With Multiplicity One

Let p in $\mathcal{M}^n$ a reduced point of the foliation $\mathcal{F}_Z$. If p is a dicritical point then its blowing-up is non-singular, thus we shall consider the case p is non-dicritical. Let $\lambda_1, \dots, \lambda_s$ be the eigenvalues of the linear part of

$DZ(p)$, then the characteristic polynomial of $M = DZ(p)$ is

$$\Delta(t) = \prod_{k=1}^{s} (t - \lambda_k)^{r_k} \tag{3.1}$$

where $\sum_{k=1}^{s} r_k = n$.

Thus, there exists $z = (z_1, \ldots, z_n)$ local coordinates of a neighborhood of p in $\mathcal{M}^n$ such that $p = (0, \ldots, 0) \in \mathbb{C}^n$ and M has Jordan canonical form:

$$M = \mathrm{diag}[M_1 \cdots M_s] \tag{3.2}$$

where $M_k \in \mathbb{C}^{r_k \times r_k}$ is the Jordan block belonging to the eigenvalue λ_k i.e.:

$$M_k = \begin{pmatrix} \lambda_k & \epsilon_1^{(k)} & 0 & \ldots & 0 & 0 \\ 0 & \lambda_k & \epsilon_2^{(k)} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda_k & \epsilon_{r_k-1}^{(k)} \\ 0 & 0 & 0 & \ldots & 0 & \lambda_k \end{pmatrix} \tag{3.3}$$

where $\epsilon_i^{(k)} \in \{0, 1\}, 1 \leq i \leq r_k - 1$ and $1 \leq k \leq s$.

A necessary and sufficient condition for $\mathcal{F}_Z$ has isolated singularities is that $\epsilon_i^{(k)} = 1$, $\forall 1 \leq i \leq r_k - 1$ and $1 \leq k \leq s$. More specifically, we have the following result:

Proposition 2. *Let $p \in \mathcal{M}^n$ be a non-dicritical, reduced singular point of the foliation $\mathcal{F}_Z$. The following assertions are equivalent:*

$$\begin{aligned} &a)\#\,\mathrm{Sing}(\tilde{F}_Z) < \infty. \\ &b) DZ(0) = \mathrm{diag}[M(\lambda_1) \ldots M(\lambda_s)] \end{aligned} \tag{3.4}$$

where $M(\lambda_k) = \lambda_k I + R_{r_k}(1)$, $\forall k = 1, \ldots, s$. (Here $I \in \mathbb{C}^{r_k \times r_k}$ is the identity matrix and $R_{r_k}(1) \in \mathbb{C}^{r_k \times r_k}$ is the upper triangular matrix of order one).

Proof.

$a) \Rightarrow b)$ In the chart $z = (z_1, \ldots, z_n)$ above, $\mathcal{F}_Z$ is generated by the vector field

$$Z = \sum_{i=1}^{n} \left(\sum_{k \geq 1} A_k^i \right) \frac{\partial}{\partial z_i}$$

By (3.2) and (3.3) we have that:

$$\begin{aligned} A_1^i(z) &= \lambda_l z_i + \epsilon_{i-t_{l-1}}^{(l)} z_{i+1},\ t_{l-1}+1 \le i \le t_l - 1,\ 1 \le l \le s \\ A_1^{t_l}(z) &= \lambda_l z_{t_l},\ 1 \le l \le s \end{aligned} \tag{3.5}$$

where $t_0 = 0$ and

$$t_l = \sum_{k=1}^{l} r_k, 1 \le l \le s.$$

Suppose by contradiction that there exists $i_0 \in \{1, \ldots, r_1 - 1\}$ such that $\epsilon_{i_0}^{(1)} = 0$. In the chart $y_1 = z_1$, $y_i = z_i/z_2 (2 \le i \le n)$, we have that:

$$\begin{aligned} \tilde{Z}(0, y_2, \ldots, y_n) &= \sum_{i=2}^{n} [A_1^i(\hat{y}) - y_i A_1^1(\hat{y})] \frac{\partial}{\partial y_i} \\ &= \sum_{i=2}^{r_1-1} [A_1^i(\hat{y}) - y_i A_1^1(\hat{y})] \frac{\partial}{\partial y_i} + [A_1^{r_1}(\hat{y}) - y_{r_1} A_1^1(\hat{y})] \frac{\partial}{\partial y_{r_1}} \\ &\quad + \sum_{l=2}^{s} \left\{ \sum_{i=t_{l-1}+1}^{t_l-1} [A_1^i(\hat{y}) - y_i A_1^1(\hat{y})] \frac{\partial}{\partial y_i} \right. \\ &\quad \left. + [A_1^{t_l}(\hat{y}) - y_{t_l} A_1^1(\hat{y})] \frac{\partial}{\partial y_{t_l}} \right\} \end{aligned}$$

where $\hat{y} = (1, y_2, \ldots, y_n)$. From (3.5):

$$\begin{aligned} \tilde{Z}(0, y_2, \ldots, y_n) &= \sum_{i=2}^{r_1-1} [\epsilon_i^{(1)} y_{i+1} - \epsilon_1^{(1)} y_2 y_i] \frac{\partial}{\partial y_i} - [\epsilon_1^{(1)} y_2 y_{r_1}] \frac{\partial}{\partial y_{r_1}} \\ &\quad + \sum_{l=2}^{s} \sum_{i=t_{l-1}+1}^{t_l-1} [(\lambda_l - \lambda_1 - \epsilon_1^{(1)} y_2) y_i + \epsilon_{i-t_{l-1}}^{(\ell)} y_{i+1}] \frac{\partial}{\partial y_i} \\ &\quad + \sum_{l=2}^{s} [\lambda_l - \lambda_1 - \epsilon_1^{(1)} y_2] y_{t_l} \frac{\partial}{\partial y_{t_l}} \end{aligned}$$

It follows that $\tilde{Z}(0, \ldots, 0, y_{i_0+1}, 0, \ldots, 0) = 0$, thus $\# \operatorname{Sing}(\tilde{F}_Z) = \infty$, which is a contradiction. We conclude that $\epsilon_i^{(1)} = 1$, $\forall 1 \le i \le r_1 - 1$ and so $M_1 = M(\lambda_1) = \lambda_1 I + R_{r_1}(1)$.

For proving that $M_\ell = M(\lambda_l) = \lambda_l I + R_{r_l}(1), (l = 2, \ldots, s)$; we consider the chart $y_j = z_j, y_i = z_i/z_j (i = 1, \ldots, n, i \neq j)$ where $j =$

$t_{l-1}+1$ and we proceed as above.

$b) \Rightarrow a)$ By hypotheses and (3.5), we have that:

$$\begin{aligned} A_1^i(z) &= \lambda_l z_i + z_{i+1},\ t_{l-1}+1 \leq i \leq t_l - 1,\ 1 \leq l \leq s \\ A_1^{t_l}(z) &= \lambda_l z_{t_l},\ 1 \leq l \leq s \end{aligned} \tag{3.6}$$

In the chart $y_1 = z_1, y_i = z_i/z_1 (2 \leq i \leq n)$, from (3.6) we obtain:

$$\begin{aligned} \tilde{Z}(0, y_2, \dots, y_n) = &\sum_{i=2}^{r_1-1} [y_{i+1} - y_2 y_i] \frac{\partial}{\partial y_i} - y_2 y_{r_1} \frac{\partial}{\partial y_{r_1}} \\ &+ \sum_{l=2}^{s} \left\{ \sum_{i=t_{l-1}+1}^{t_l - 1} [(\lambda_l - \lambda_1 - y_2) y_i + y_{i+1}] \frac{\partial}{\partial y_i} \right. \\ &\left. + [\lambda_l - \lambda_1 - y_2] y_{t_l} \frac{\partial}{\partial y_{t_l}} \right\} \end{aligned} \tag{3.7}$$

It follows that $(0, \dots, 0)$ is the unique singularity of $\tilde{F}_Z$ in this chart. Now, for $i_0 \in \{2, \dots, r_1 - 1\}$ (respectively $i_0 = r_1$), we consider the chart $y_{i_0} = z_{i_0}$, $y_i = z_i/z_{i_0}$, $1 \leq i \leq n$, $i \neq i_0$ (respectively $y_{r_1} = z_{r_1}$, $y_i = z_i/z_{r_1}$, $1 \leq i \leq n$, $i \neq r_1$).

Denoting

$$y_0 = (y_1, \dots, y_{i_0-1}, 0, y_{i_0+1}, \dots, y_n)$$

and

$$\hat{y} = (y_1, \dots, y_{i_0-1}, 1, y_{i_0+1}, \dots, y_n)$$

respectively

$$y_0 = (y_1, \dots, y_{r_1-1}, 0, y_{r_1+1}, \dots, y_n$$

and

$$\hat{y} = (y_1, \dots, y_{r_1-1}, 1, y_{r_1+1}, \dots, y_n),$$

from (1.9) and (3.7), we have that:

$$
\begin{aligned}
\tilde{Z}(y_0) &= \sum_{\substack{i=1\\ i\neq i_0}}^{n} [A_1^i(\hat{y}) - y_i A_1^{i_0}(\hat{y})]\frac{\partial}{\partial y_i} \\
&= \sum_{\substack{i=1\\ i\neq i_0-1}}^{r_1-1} [y_{i+1} - y_i y_{i_0+1}]\frac{\partial}{\partial y_i} \\
&\quad + [1 - y_{i_0-1}y_{i_0+1}]\frac{\partial}{\partial y_{i_0-1}} - y_{i_0+1}y_{r_1}\frac{\partial}{\partial y_{r_1}} \qquad (3.8)\\
&\quad + \sum_{l=2}^{s}\Bigg\{ \sum_{i=t_{l-1}+1}^{t_l-1} [(\lambda_l - \lambda_1 - y_{i_0+1})y_i + y_{i+1}]\frac{\partial}{\partial y_i} \\
&\quad + [\lambda_l - \lambda_1 - y_{i_0+1}]y_{t_l}\frac{\partial}{\partial y_{t_l}}\Bigg\}
\end{aligned}
$$

(respectively)

$$
\begin{aligned}
\tilde{Z}(y_0) &= \sum_{\substack{i=1\\ i\neq r_1}}^{n} [A_1^i(\hat{y}) - y_i A_1^{r_1}(\hat{y})]\frac{\partial}{\partial y_i} \\
&= \sum_{\substack{i=1\\ i\neq r_1}}^{r_1-1} y_{i+1}\frac{\partial}{\partial y_i} + \frac{\partial}{\partial y_{r_1-1}} \qquad (3.9)\\
&\quad + \sum_{l=2}^{s}\sum_{i=t_{l-1}+1}^{t_l-1} [(\lambda_l - \lambda_1)y_i + y_{i+1}]\frac{\partial}{\partial y_i} \\
&\quad + \sum_{l=2}^{s}[\lambda_l - \lambda_1]y_{t_l}\frac{\partial}{\partial y_{t_l}}
\end{aligned}
$$

From (3.8) and (3.9), it follows that $\tilde{\mathcal{F}}_Z$ has not singularities in these charts.

Similarly, we can prove that $\mathrm{Sing}(\tilde{\mathcal{F}}_Z) = \{\tilde{p}_1, \ldots, \tilde{p}_s\}$, where $\tilde{p}_l$ is the zero at the chart $y_j = z_j$, $y_i = z_i/z_j$, $i = 1, \ldots, n$, $i \neq j$, $1 \leq l \leq s$ and $j = t_{l-1} + 1$. This finishes the proof of Proposition 2.

Remark. The points $\tilde{p}_1, \ldots, \tilde{p}_s$ above are non-dicritical singularities of $\tilde{\mathcal{F}}_Z$.

Now, we consider the linear part of $\tilde{Z}$ at each non-dicritical singular point $\tilde{p}_1, \dots, \tilde{p}_s$. In the chart $y_1 = z_1$, $y_i = z_i/z_1 (2 \leq i \leq n)$, it is not difficult to see that:

$$D\tilde{Z}(0) = \begin{pmatrix} M_1 & \Theta & \dots & \Theta \\ P_1 & M(\lambda_2 - \lambda_1) & \dots & \Theta \\ \vdots & \vdots & & \vdots \\ P_{s-1} & \Theta & \dots & M(\lambda_s - \lambda_1) \end{pmatrix} \tag{3.10}$$

where $M_1 \in \mathbb{C}^{r_1 \times r_1}$, $M(\lambda_l - \lambda_1) \in \mathbb{C}^{r_l \times r_l}$ and $P_{l-1} \in \mathbb{C}^{r_l \times r_1} (l = 2, \dots, s)$ are defined as

$$M_1 = \begin{pmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ \alpha_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{r_1 - 1} & 0 & 0 & \dots & 1 \\ \alpha_{r_1} & 0 & 0 & \dots & 0 \end{pmatrix} P_{l-1} = \begin{pmatrix} \alpha_{t_{l-1}+1} & 0 & \dots & 0 \\ \alpha_{t_{l-1}+2} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \alpha_{t_l} & 0 & \dots & 0 \end{pmatrix} \tag{3.11}$$

and $M(\lambda_l - \lambda_1) = (\lambda_l - \lambda_1)I + R_{r_\ell}(1)$. (Here $\alpha_i = A_2^i(1, 0, \dots, 0)$, $2 \leq i \leq n$.) Notice that we have three possibilities for characteristic polynomial $\tilde{\Delta}(t)$ of $\tilde{M} = D\tilde{Z}(0)$:

a) If $\lambda_1 = 0$ then

$$\tilde{\Delta}(t) = t^{r_1} \prod_{l=2}^{s} (t - \lambda_l)^{r_l} \tag{3.12}$$

b) If $\lambda_l \neq 2\lambda_1$, $\forall l = 2, \dots, s$ then

$$\tilde{\Delta}(t) = t^{r_1 - 1}(t - \lambda_1) \prod_{l=2}^{s} (t - \lambda_l + \lambda_1)^{r_l} \tag{3.13}$$

c) If there exists $l_0 \in \{2, \dots, s\}$ such that $\lambda_1 = 2\lambda_l$ then we can suppose, without loss of generality, than $l_0 = 2$ and so

$$\tilde{\Delta}(t) = t^{r_1 - 1}(t - \lambda_1)^{r_2 + 1} \prod_{l=3}^{s} (t - \lambda_l + \lambda_1)^{r_l} \tag{3.14}$$

Now, if we will suppose that $\tilde{p}_1$ satisfies $\# \operatorname{Sing}(\mathcal{F}_Z^{(2)}) < \infty$ where $\mathcal{F}_Z^{(2)} = E_2^* \tilde{\mathcal{F}}_Z$ and E_2 is the blowing-up with center at $\tilde{p}_1$, then $\tilde{M}$ is a matrix of type (3.4). More specifically, denoting:

$$[\lambda_1, \dots, \lambda_s; r_1, \dots, r_s] = \operatorname{diag}[M(\lambda_1) \cdots M(\lambda_s)] \tag{3.15}$$

we have the following:

Proposition 3. *Let* $\tilde{p}_1 \in \mathrm{Sing}(\mathcal{F}_Z)$ *such that* $\#\,\mathrm{Sing}(\mathcal{F}_Z^{(2)}) < \infty$, *then*

$$\tilde{M} = \begin{cases} [0, \lambda_2, \dots, \lambda_s; r_1, r_2, \dots, r_s], \\ \quad \text{if } \lambda_1 = 0 \\ [0, \lambda_1, \lambda_2 - \lambda_1, \dots, \lambda_s - \lambda_1; r_1 - 1, 1, r_2, \dots, r_s], \\ \quad \text{if } \lambda_1 \neq 0 \text{ and } \lambda_l \neq 2\lambda_1 \\ [0, \lambda_1, \lambda_3 - \lambda_1, \dots, \lambda_s - \lambda_1; r_1 - 1, r_2 + 1, r_3, \dots, r_s], \\ \quad \text{if } \lambda_1 \neq 0 \text{ and } \lambda_2 = 2\lambda_1 \end{cases}$$

Proof. By hypotheses and Proposition 2, the minimum polynomial $\tilde{m}(t)$ of $\tilde{M}$ is $\tilde{m}(t) = \tilde{\Delta}(t)$. Now, considering the cases a), b) and c) above, the proof it follows.

Remark. A similar result is obtained for the other singular points $\tilde{p}_2, \dots \tilde{p}_s \in \mathrm{Sing}(\mathcal{F}_Z)$.

Now, we can assert that if $p \in \mathcal{M}^n$ is a reduced, non-dicritical singular points of the foliation $\mathcal{F}_Z$ such that p is an A.I.S., then do not appear dicritical points in the blowing-up process. In fact, by Proposition 2, any point of $\mathrm{Sing}(\mathcal{F}_Z)$ is a non-dicritical point and by Proposition 3, the linear part of $\tilde{Z}$ is similar to the linear part of Z. So, the proof it follows by induction.

In [**C-C-S**], the authors study the final forms of an absolutely isolated non-dicritical singularity. Since, if $p \in \mathcal{M}^n$ is a reduced, non-dicritical singular point of the foliation $\mathcal{F}_Z$ such that p is an A.I.S., then p is an absolutely isolated non-dicritical singularity of the foliation $\mathcal{F}_Z$, and so, we can apply the results in [**C-C-S**].

References

[**B**] I. Bendixson: *Sur les points singuliers des équations différentielles*, Ofv. Kongl. Vetenskaps Akademiens Förhandlinger, Stokholm, Vol. 9, 186 (1898), p. 635–658.

[**C-C-S**] C. Camacho, F. Cano, P. Sad: *Absolutely isolated singularities of holomorphic vector fields*, Invent. math. 98 (1989), p. 351–369.

[**C-L-S**] C. Camacho, A. Lins Neto, P. Sad: *Topological invariants and equidesingularization for holomorphic vector fields*, J. Differ. Geom. 20 (1984), p. 143–174.

[**C-S**] C. Camacho, P. Sad: *Pontos singulares de Equações Diferenciais Analíticas*, 16° Colóquio Brasileiro de Matemática, IMPA, (1987).

[**C**] C. Camacho: *On the local structure of conformal mappings and holomorphic vector fields in* $\mathbb{C}^2$, Asterisque 59–60 (1978), p. 83–94.

[**Ca1**] F. Cano: *Final forms for a 3-dimensional vector field under blowing-up*, Ann. Inst. Fourier 3, 2 (1987), p. 151–193.

[**Ca2**] F. Cano: *Desingularization strategies for 3-dimensional vector fields*, Lecture Notes in Math., Vol. 1259. Berlin, Heidelberg, New York; Springer (1987).

[**Ch**] E. Chirka: *Complex Analytic Sets*, MIA, Kluwer Academic Publishers. Dordrecht, Boston, London (1989).

[**D**] H. Dulac: *Recherches sur les points singuliers des équations différentielles*, J. École polytechnique, Vol. 2, sec. 9 (1904), p. 1–125.

[**G-H**] P. Griffiths, J. Harris: *Principles of Algebraic Geometry*, Willey-Interscience, New York (1978).

[**M-M**] J. Mattei, R. Moussu: *Holonomie et intégrales premières*, Ann. Sci. Ecole. Norm. Sup. (4) 13 (1980), p. 469–523.

[**S**] A. Seidenberg: *Reduction of singularities of the differentiable equation $Ady = Bdx$*, Amer. J. Math. 90 (1968), p. 248–269.

[**V**] A. Ven Den Essen: *Reduction of singularities of the differentiable equation $Ady = Bdx$*, Lecture Notes in Math., Vol. 712, p. 44–59, Springer-Verlag.

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